

On pre-Hamiltonian Cycles in Hamiltonian Digraphs

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Abstract

Let D be a strongly connected directed graph of order $n \geq 4$. In [14] (J. of Graph Theory, Vol.16, No. 5, 51-59, 1992) Y. Manoussakis proved the following theorem: Suppose that D satisfies the following condition for every triple x, y, z of vertices such that x and y are non-adjacent: If there is no arc from x to z , then $d(x) + d(y) + d^+(x) + d^-(z) \geq 3n - 2$. If there is no arc from z to x , then $d(x) + d(y) + d^-(x) + d^+(z) \geq 3n - 2$. Then D is Hamiltonian. In this paper we show that: If D satisfies the condition of Manoussakis' theorem, then D contains a pre-Hamiltonian cycle (i.e., a cycle of length $n - 1$) or n is even and D is isomorphic to the complete bipartite digraph with partite sets of cardinalities $n/2$ and $n/2$.

Keywords: Digraph, cycles, Hamiltonian cycles, pre-Hamiltonian cycles, longest non-Hamiltonian cycles.

1 Introduction

The directed graph (digraph) D is Hamiltonian if it contains a Hamiltonian cycle, i.e., a cycle of length n and is pancyclic if it contains cycles of all lengths m , $3 \leq m \leq n$, where n is the number of vertices in D . We recall the following well-known degree conditions (Theorems 1.1-1.8) that guarantee that a digraph is Hamiltonian. In each of the conditions (Theorems 1.1-1.8) below D is a strongly connected digraph of order n :

Theorem 1.1 (Ghouila-Houri [12]). If $d(x) \geq n$ for all vertices $x \in V(D)$, then D is Hamiltonian.

Theorem 1.2 (Woodall [18]). If $d^+(x) + d^-(y) \geq n$ for all pairs of vertices x and y such that there is no arc from x to y , then D is Hamiltonian.

Theorem 1.3 (Meyniel [15]). If $n \geq 2$ and $d(x) + d(y) \geq 2n - 1$ for all pairs of non-adjacent vertices in D , then D is Hamiltonian .

It is easy to see that Meyniel's theorem is a common generalization of Ghouila-Houri's and Woodall's theorems. For a short proof of Theorem 1.3, see [5].

C. Thomassen [17] (for $n = 2k + 1$) and S. Darbinyan [7] (for $n = 2k$) proved the following:

Theorem 1.4. If D is a digraph of order $n \geq 5$ with minimum degree at least $n - 1$ and with minimum semi-degree at least $n/2 - 1$, then D is Hamiltonian (unless some extremal cases which are characterized).

For the next theorem we need the following:

Definition 1 [14]. Let k be an arbitrary nonnegative integer. A digraph D satisfies the condition A_k if and only if for every triple x, y, z of vertices such that x and y are non-adjacent: If there is no arc from x to z , then $d(x) + d(y) + d^+(x) + d^-(z) \geq 3n - 2 + k$. If there is no arc from z to x , then

$$d(x) + d(y) + d^-(x) + d^+(z) \geq 3n - 2 + k.$$

Theorem 1.5 (Y. Manoussakis [14]). If a digraph D satisfies the condition A_0 , then D is Hamiltonian.

Each of these theorems imposes a degree condition on all pairs of non-adjacent vertices (or on all vertices). In the following three theorems imposes a degree condition only for some pairs of non-adjacent vertices.

Theorem 1.6 (Bang-Jensen, Gutin, H.Li [2]). Suppose that $\min\{d(x), d(y)\} \geq n-1$ and $d(x)+d(y) \geq 2n-1$ for any pair of non-adjacent vertices x, y with a common in-neighbour, then D is Hamiltonian.

Theorem 1.7 (Bang-Jensen, Gutin, H.Li [2]). Suppose that $\min\{d^+(x) + d^-(y), d^-(x) + d^+(y)\} \geq n$ for any pair of non-adjacent vertices x, y with a common out-neighbour or a common in-neighbour, then D is Hamiltonian.

Theorem 1.8 (Bang-Jensen, Guo, Yeo [3]). Suppose that $d(x) + d(y) \geq 2n-1$ and $\min\{d^+(x) + d^-(y), d^-(x) + d^+(y)\} \geq n-1$ for any pair of non-adjacent vertices x, y with a common out-neighbour or a common in-neighbour, then D is Hamiltonian.

Note that Theorem 1.8 generalizes Theorem 1.7.

In [11, 16, 6, 8] it was shown that if a digraph D satisfies the condition one of Theorems 1.1, 1.2, 1.3 and 1.4, respectively, then D also is pancyclic (unless some extremal cases which are characterized). It is natural to set the following problem: Characterize those digraphs which satisfy the conditions of Theorem 1.6 (1.7, 1.8) but are not pancyclic. In the many papers (as well as, in the mentioned papers), the existence of a pre-Hamiltonian cycle (i.e., a cycle of length $n-1$) is essential to show that a given digraph (graph) is pancyclic or not. This indicates that the existence of a pre-Hamiltonian cycle in the a digraph (graph) makes the pancyclic problem significantly easier, in a sense. In [9] the following results were proved:

- (i) if the minimum semi-degree of D at least two and D satisfies the condition of Theorem 1.6 or
- (ii) D is not directed cycle and satisfies the condition of Theorem 1.7, then either D contains a pre-Hamiltonian cycle (i.e., a cycle of length $n-1$) or n is even and D is isomorphic to the complete bipartite digraph or to the complete bipartite digraph minus one arc with partite sets of cardinalities $n/2$ and $n/2$.

In [10] proved that if D is not a directed cycle and satisfies the condition of Theorem 1.8, then D contains a pre-Hamiltonian cycle or a cycle of length $n-2$.

In [14] the following conjecture was proposed:

Conjecture 1.9. Any strongly connected digraph satisfying the condition A_3 is pancyclic.

In this paper using some claims of the proof of Theorem 1.5 (see [14]) we prove the following:

Theorem 1.10. Any strongly connected digraph D on $n \geq 4$ vertices satisfying the condition A_0 contains a pre-Hamiltonian cycle or n is even and D is isomorphic to the complete bipartite digraph with partite sets of cardinalities of $n/2$ and $n/2$.

The following examples show the sharpness of the bound $3n-2$ in the theorem. The digraph consisting of the disjoint union of two complete digraphs with one common vertex and the digraph obtained from

a complete bipartite digraph after deleting one arc show that the bound $3n - 2$ in the above theorem is best possible.

2 Terminology and Notations

We shall assume that the reader is familiar with the standard terminology on the directed graphs (digraph) and refer the reader to the monograph of Bang-Jensen and Gutin [1] for terminology not discussed here. In this paper we consider finite digraphs without loops and multiple arcs. For a digraph D , we denote by $V(D)$ the vertex set of D and by $A(D)$ the set of arcs in D . The order of D is the number of its vertices. Often we will write D instead of $A(D)$ and $V(D)$. The arc of a digraph D directed from x to y is denoted by xy . For disjoint subsets A and B of $V(D)$ we define $A(A \rightarrow B)$ as the set $\{xy \in A(D)/x \in A, y \in B\}$ and $A(A, B) = A(A \rightarrow B) \cup A(B \rightarrow A)$. If $x \in V(D)$ and $A = \{x\}$ we write x instead of $\{x\}$. If A and B are two disjoint subsets of $V(D)$ such that every vertex of A dominates every vertex of B , then we say that A dominates B , denoted by $A \rightarrow B$. The out-neighborhood of a vertex x is the set $N^+(x) = \{y \in V(D)/xy \in A(D)\}$ and $N^-(x) = \{y \in V(D)/yx \in A(D)\}$ is the in-neighborhood of x . Similarly, if $A \subseteq V(D)$, then $N^+(x, A) = \{y \in A/xy \in A(D)\}$ and $N^-(x, A) = \{y \in A/yx \in A(D)\}$. The out-degree of x is $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ is the in-degree of x . Similarly, $d^+(x, A) = |N^+(x, A)|$ and $d^-(x, A) = |N^-(x, A)|$. The degree of the vertex x in D is defined as $d(x) = d^+(x) + d^-(x)$ (similarly, $d(x, A) = d^+(x, A) + d^-(x, A)$). The subdigraph of D induced by a subset A of $V(D)$ is denoted by $\langle A \rangle$. The path (respectively, the cycle) consisting of the distinct vertices x_1, x_2, \dots, x_m ($m \geq 2$) and the arcs $x_i x_{i+1}$, $i \in [1, m-1]$ (respectively, $x_i x_{i+1}$, $i \in [1, m-1]$, and $x_m x_1$), is denoted by $x_1 x_2 \cdots x_m$ (respectively, $x_1 x_2 \cdots x_m x_1$). We say that $x_1 x_2 \cdots x_m$ is a path from x_1 to x_m or is an (x_1, x_m) -path. For a cycle $C_k := x_1 x_2 \cdots x_k x_1$ of length k , the subscripts considered modulo k , i.e., $x_i = x_s$ for every s and i such that $i \equiv s \pmod{k}$. A cycle that contains the all vertices of D (respectively, the all vertices of D except one) is a Hamiltonian cycle (respectively, is a pre-Hamiltonian cycle). The concept of the pre-Hamiltonian cycle was given in [13]. If P is a path containing a subpath from x to y we let $P[x, y]$ denote that subpath. Similarly, if C is a cycle containing vertices x and y , $C[x, y]$ denotes the subpath of C from x to y . A digraph D is strongly connected (or, just, strong) if there exists a path from x to y and a path from y to x for every pair of distinct vertices x, y . For an undirected graph G , we denote by G^* the symmetric digraph obtained from G by replacing every edge xy with the pair xy, yx of arcs. $K_{p,q}$ denotes the complete bipartite graph with partite sets of cardinalities p and q . Two distinct vertices x and y are adjacent if $xy \in A(D)$ or $yx \in A(D)$ (or both). For integers a and b , $a \leq b$, let $[a, b]$ denote the set of all integers which are not less than a and are not greater than b . Let C be a non-Hamiltonian cycle in digraph D . An (x, y) -path P is a C -bypass if $|V(P)| \geq 3$, $x \neq y$ and $V(P) \cap V(C) = \{x, y\}$.

3 Preliminaries

The following well-known simple Lemmas 3.1-3.4 are the basis of our results and other theorems on directed cycles and paths in digraphs. They will be used extensively in the proofs of our results.

Lemma 3.1 [11]. Let D be a digraph of order $n \geq 3$ containing a cycle C_m , $m \in [2, n-1]$. Let x be a vertex not contained in this cycle. If $d(x, C_m) \geq m+1$, then D contains a cycle C_k for all $k \in [2, m+1]$.

The following lemma is a slight modification of a lemma by Bondy and Tomassen [5].

Lemma 3.2. Let D be a digraph of order $n \geq 3$ containing a path $P := x_1x_2 \dots x_m$, $m \in [2, n-1]$ and let x be a vertex not contained in this path. If one of the following conditions holds:

- (i) $d(x, P) \geq m+2$;
- (ii) $d(x, P) \geq m+1$ and $xx_1 \notin D$ or $x_mx_1 \notin D$;
- (iii) $d(x, P) \geq m$, $xx_1 \notin D$ and $x_mx_1 \notin D$, then there is an $i \in [1, m-1]$ such that $x_ix, xx_{i+1} \in D$, i.e., D contains a path $x_1x_2 \dots x_ixx_{i+1} \dots x_m$ of length m (we say that x can be inserted into P or the path $x_1x_2 \dots x_ixx_{i+1} \dots x_m$ is extended from P with x). \square

If in Lemma 3.1 and Lemma 3.2 instead of the vertex x consider a path Q , then we get the following Lemmas 3.3 and 3.4, respectively.

Lemma 3.3. Let $C_k := x_1x_2 \dots x_kx_1$, $k \geq 2$, be a non-Hamiltonian cycle in a digraph D . Moreover, assume that there exists a path $Q := y_1y_2 \dots y_r$, $r \geq 1$, in $D - C_k$. If $d^-(y_1, C_k) + d^+(y_r, C_k) \geq k+1$, then for all $m \in [r+1, k+r]$ the digraph D contains a cycle C_m of length m with vertex set $V(C_m) \subseteq V(C_k) \cup V(Q)$. \square

Lemma 3.4. Let $P := x_1x_2 \dots x_k$, $k \geq 2$, be a non-Hamiltonian path in a digraph D . Moreover, assume that there exists a path $Q := y_1y_2 \dots y_r$, $r \geq 1$, in $D - P$. If $d^-(y_1, P) + d^+(y_r, P) \geq k + d^-(y_1, \{x_k\}) + d^+(y_r, \{x_1\})$, then D contains a path from x_1 to x_k with vertex set $V(P) \cup V(Q)$. \square

For the proof of our result we also need the following

Lemma 3.5 [14]. Let D be a digraph on $n \geq 3$ vertices satisfying the condition A_0 . Assume that there are two distinct pairs of non-adjacent vertices x, y and x, z in D . Then either $d(x) + d(y) \geq 2n-1$ or $d(x) + d(z) \geq 2n-1$.

4 The proof of Theorem 1.10

In the proof of Theorem 1.10 we often will use the following definition:

Definition 2. Let $P_0 := x_1x_2 \dots x_m$, $m \geq 2$, be an arbitrary (x_1, x_m) -path in a digraph D and let $y_1, y_2, \dots, y_k \in V(D) - V(P_0)$. For $i \in [1, k]$ we denote by P_i an (x_1, x_m) -path in D with vertex set $V(P_{i-1}) \cup \{y_i\}$ (if it exists), i.e., P_i is extended path obtained from P_{i-1} with some vertex y_i , where $y_i \notin V(P_{i-1})$. If $e+1$ is the maximum possible number of these paths P_0, P_1, \dots, P_e , $e \in [0, k]$, then we say that P_e is extended path obtained from P_0 with vertices y_1, y_2, \dots, y_k as much as possible. Notice that P_i ($i \in [0, e]$) is an (x_1, x_m) -path of length $m+i-1$.

Proof of Theorem 1.10. Let $C := x_1x_2 \dots x_kx_1$ be a longest non-Hamiltonian cycle in D of length k , and let C be chosen so that $\langle V(D) - V(C) \rangle$ has the minimum number of connected components. Suppose that $k \leq n-2$ and $n \geq 5$ (the case $n=4$ is trivial). It is easy to show that $k \geq 3$. We will prove that D is isomorphic to the complete bipartite digraph $K_{n/2, n/2}^*$. Put $R := V(D) - V(C)$. Let R_1, R_2, \dots, R_q be the connected components of $\langle R \rangle$ (i.e., if $q \geq 2$, then for any pair i, j , $i \neq j$, there is no arc between R_i and R_j). In [14] it was proved that for any R_i , $i \in [1, q]$, the subdigraph $\langle V(C) \cup V(R_i) \rangle$ contains a C -bypass. (The existence of a C -bypass also follows from Bypass Lemma (see [4]), since $\langle V(C) \cup V(R_i) \rangle$ is strong and condition A_0 implies that the underlying graph of the subdigraph $\langle V(C) \cup V(R_i) \rangle$ is 2-connected). Let $P := x_my_1y_2 \dots y_{t_i}x_{m+\lambda_i}$ be a C -bypass in $\langle V(C) \cup V(R_i) \rangle$ ($i \in [1, q]$ is arbitrary) and λ_i is considered to be minimum in the sense that there is no C -bypass $x_au_1u_2 \dots u_{l_i}x_{a+r_i}$ in $\langle V(C) \cup V(R_i) \rangle$ such that $r_i < \lambda_i$ and $\{x_a, x_{a+r_i}\}$ is a subset of $\{x_m, x_{m+1}, \dots, x_{m+\lambda_i}\}$.

We will distinguish two cases, according as there is a λ_i , $i \in [1, q]$, such that $\lambda_i = 1$ or not.

Assume first that $\lambda_i \geq 2$ for all $i \in [1, q]$. For this case one can show that (the proofs as the same as the proofs of Case 1, Lemma 2.3 and Claim 1 in [14]) if $\lambda_i \geq 2$, then $t_i = |R_i| = 1$, in $\langle V(C) \rangle$ there is an $(x_{m+\lambda_i}, x_m)$ -path (say, P') of length $k - 2$ with vertex set $V(P') = V(C) - \{z_i\}$, where $z_i \in \{x_{m+1}, x_{m+2}, \dots, x_{m+\lambda_i-1}\}$ and $d(y_1) + d(z_i) \leq 2n - 2$ (note that y_1 and z_i are non-adjacent). From $|R| \geq 2$ and $|R_i| = 1$ (for all i) it follows that $q \geq 2$. If $u \in R_2$, then $d(u) = d(u, C) \leq k$ (by Lemma 3.1) and $d(z_1, R) = 0$ (by minimality of q), in particular, the vertices z_1 and u are non-adjacent. Therefore $d(z_1) = d(z_1, C) \leq k$ and $d(z_1) + d(u) \leq 2n - 2$. This in connection with $d(y_1) + d(z_1) \leq 2n - 2$ contradicts Lemma 3.5.

Assume second that $\lambda_i = 1$ for all $i \in [1, q]$. It is clear that $q = 1$. Put $t := t_1$ and $\lambda := \lambda_1 = 1$. Now for this case first we will prove Claims 1-15.

Observe that if $v_1 v_2 \dots v_j$ (maybe, $j = 1$) is a path in $\langle R \rangle$ and $x_i v_1 \in D$, then $v_j x_{i+j} \notin D$ since C is longest non-Hamiltonian cycle in D . We shall use this often, without mentioning this explicitly.

From $\lambda = 1$ and the maximality of C it follows the following:

Claim 1. $R = \{y_1, y_2, \dots, y_t\}$, i.e., $t = n - k \geq 2$ and $y_1 y_2 \dots y_t$ is a Hamiltonian path in $\langle R \rangle$, and if $1 \leq i < j - 1 \leq t - 1$, then $y_i y_j \notin D$. \square

Claim 1 implies that

$$d^+(y_1, R) = d^-(y_t, R) = 1 \quad \text{and if } i \in [1, t - 1], \text{ then } d^+(y_i) \leq i; \quad (1)$$

$$d(y_1, R), d(y_t, R) \leq n - k \quad \text{and if } i \in [2, t - 1], \text{ then } d(y_i, R) \leq n - k + 1. \quad (2)$$

Claim 2. (i). If $x_i y_1 \in D$, then $d^-(x_{i+1}, \{y_1, y_2, \dots, y_{t-1}\}) = 0$;

(ii). If $y_t x_{i+1} \in D$, then $d^+(x_i, \{y_2, y_3, \dots, y_t\}) = d^+(x_{i-1}, \{y_1, y_2, \dots, y_{t-1}\}) = 0$;

(iii). $d(y_j, C) \leq k$ for all $j \in [1, t]$ and in addition, if $x_i y_1$ and $y_t x_{i+1} \in D$, then $d(y_j, C) \leq k - 1$ for all $j \in [2, t - 1]$ (by Lemma 3.2(iii) and Claim 2(ii)). \square

Claim 3. Assume that $\langle R \rangle$ is strong. Then there are no two distinct vertices x_i, x_j ($i, j \in [1, k]$) such that $d^+(x_i, R) \geq 1$, $d^-(x_j, R) \geq 1$, $|C[x_i, x_j]| \geq 3$, $d^-(x_{j-1}, R) = 0$ (respectively, $d^+(x_{i+1}, R) = 0$), moreover if $|C[x_i, x_j]| \geq 4$, then $A(R, C[x_{i+1}, x_{j-2}]) = \emptyset$ (respectively, $A(R, C[x_{i+2}, x_{j-1}]) = \emptyset$).

Proof. Suppose that Claim 3 is false. Without loss of generality assume that $x_k y_f, y_g x_l \in D$ ($l \in [2, k - 1]$) $d^-(x_{l-1}, R) = 0$ and if $l \geq 3$, then $A(R, C[x_1, x_{l-2}]) = \emptyset$. The subdigraph $\langle R \rangle$ contains a (y_f, y_g) -path (say $P(y_f, y_g)$) since R is strong. We extend the path $P_0 := C[x_l, x_k]$ with the vertices x_1, x_2, \dots, x_{l-1} as much as possible. Then some vertices $z_1, z_2, \dots, x_d \in \{x_1, x_2, \dots, x_{l-1}\}$, $d \in [1, l - 1]$, are not on the extended path P_e (for otherwise, it is not difficult to see that by Definition 2 there is an (x_l, x_k) -path P_i , $i \in [0, e]$, which together with the path $P(y_f, y_g)$ and the arcs $x_k y_f, y_g x_l$ forms a non-Hamiltonian cycle longer than C). Therefore, by Lemma 3.2(i), for all $s \in [1, d]$ the following holds

$$d(z_s, C) \leq k + d - 1. \quad (3)$$

From $A(R, C[x_1, x_{l-2}]) = \emptyset$ (if $l \geq 3$), $d^-(x_{l-1}, R) = 0$ and Lemma 3.2(ii) it follows that

$$d(y_1, C) \leq k - l + 2 \quad \text{and} \quad d(y_t, C) \leq k - l + 2$$

since neither y_1 nor y_t cannot be inserted into $C[x_{l-1}, x_k]$. This together with (2) implies that

$$d(y_1) \leq n - l + 2 \quad \text{and} \quad d(y_t) \leq n - l + 2. \quad (4)$$

If there exists a z_s such that $d(z_s, R) = 0$, then by (3) and (4) we obtain that $d(z_s) + d(y_1) \leq 2n - 2$ and $d(z_s) + d(y_t) \leq 2n - 2$, which contradicts Lemma 3.5. Assume therefore that there is no z_s such

that $d(z_s, R) = 0$. Then $d = 1$, $z_1 = x_{l-1}$, $d^+(x_{l-1}, R) \geq 1$, $d(x_{l-1}, C) \leq k$ (by (3)) and D contains an (x_l, x_k) -path with vertex set $V(C) - \{x_{l-1}\}$. From this it follows that $y_f = y_g$, i.e.,

$$d^+(x_k, R - \{y_f\}) = d^-(x_l, R - \{y_f\}) = 0. \quad (5)$$

Therefore D contains a cycle C' of length k with vertex set $V(C) \cup \{y_f\} - \{x_{l-1}\}$, and the vertices x_{l-1}, y_f are non-adjacent. From this, (3), $d^-(x_{l-1}, R) = 0$ and $d(x_{l-1}, \{y_f\}) = 0$ it follows that $d(x_{l-1}) \leq n - 1$.

Assume first that $y_f \neq y_1$. Let $x_{l-1}y_1 \in D$. Then $y_f = y_t$ (by Claim 2(i)) and for the triple of vertices y_t, x_{l-1}, y_1 condition A_0 holds, since $y_1x_{l-1} \notin D$ and y_t, x_{l-1} are non-adjacent. From $d(x_{l-1}, R - \{y_1\}) = 0$ and (3) it follows that $d(x_{l-1}) \leq k + 1$. Since D contains no cycle of length $k + 1$, it follows that for the arc $x_{l-1}y_1$ and the cycle C' , by Lemma 3.3 the following holds $d^-(x_{l-1}, C') + d^+(y_1, C') \leq k$. This together with $d^+(y_1, R) = 1$ and $d^-(x_{l-1}, R) = 0$ implies that $d^-(x_{l-1}) + d^+(y_1) \leq n - 2$ (here we consider the cases $k = n - 2$ and $k \leq n - 3$ separately). Therefore, by condition A_0 , (4), $d(x_{l-1}) \leq n - 1$, $l \geq 2$ and $k \leq n - 2$, we have

$$3n - 2 \leq d(y_t) + d(x_{l-1}) + d^-(x_{l-1}) + d^+(y_1) \leq 3n - 3,$$

a contradiction. Let now $x_{l-1}y_1 \notin D$. Then the vertices x_{l-1}, y_1 are non-adjacent and $t \geq 3$ since $d^+(x_{l-1}, R) \geq 1$. Using (2) and Lemma 3.2(iii) it is not difficult to see that $d(y_1) \leq n - l$, since $x_ky_1 \notin D$ and $y_1x_l \notin D$ (by (5)). Notice that

$$d(x_{l-1}) = d(x_{l-1}, C) + d(x_{l-1}, R - \{y_1, y_f\}) \leq k + d(x_{l-1}, R - \{y_1, y_f\}) \leq n - 2,$$

and (by Lemma 3.2(i))

$$d(y_f) = d(y_f, C) + d(y_f, R) \leq k - l + 2 + d(y_f, R).$$

From the last three inequalities we obtain that

$$d(y_1) + d(x_{l-1}) \leq 2n - 2 - l$$

and

$$d(y_f) + d(x_{l-1}) \leq 2k - l + 2 + d(x_{l-1}, R - \{y_1, y_f\}) + d(y_f, R).$$

Notice that

$$d(x_{l-1}, R - \{y_1, y_f\}) + d(y_f, R) \leq n - k - 2 + n - k = 2n - 2k - 2$$

since if $x_{l-1}y_j \in D$, then $y_jy_f \notin D$, where $y_j \neq y_1, y_f$. Therefore $d(y_f) + d(x_{l-1}) \leq 2n - l \leq 2n - 2$. This together with $d(y_1) + d(x_{l-1}) \leq 2n - 2 - l$ contradicts Lemma 3.5.

Assume next that $y_f = y_1$. If x_{l-1}, y_t are non-adjacent, then $d(x_{l-1}, R) \leq n - k - 2$ since $d(x_{l-1}, \{y_1, y_t\}) = 0$ and hence by (3) and $d = 1$, $d(x_{l-1}) \leq n - 2$. Therefore, using (4) we get that $d(y_1) + d(x_{l-1}) \leq 2n - 2$ and $d(y_t) + d(x_{l-1}) \leq 2n - 2$ which contradicts Lemma 3.5, since y_1, x_{l-1} and y_t, x_{l-1} are two distinct pairs of non-adjacent vertices. So, we can assume that $x_{l-1}y_t \in D$. Since C' is a longest non-Hamiltonian cycle, $d^-(x_{l-1}, R) = 0$ and $d^+(y_t, R - \{y_1\}) \leq n - k - 2$, from Lemma 3.3 it follows that $d^-(x_{l-1}) + d^+(y_t) \leq n - 2$. Then from (4) and $d(x_{l-1}) \leq n - 1$, by condition A_0 , for the triple of the vertices x_{l-1}, y_1, y_t we obtain that

$$3n - 2 \leq d(y_1) + d(x_{l-1}) + d^+(y_t) + d^-(x_{l-1}) \leq 3n - l - 1 \leq 3n - 3,$$

which is a contradiction. Claim 3 is proved. \square

Now we divide the proof of the theorem into two parts: $k \leq n - 3$ and $k = n - 2$.

Part 1. $k \leq n - 3$, i.e., $t \geq 3$. For this part first we will prove the following Claims 4-9 below.

Claim 4. Let $t \geq 3$ and $y_t y_1 \in D$. Then (i) if $x_i y_1 \in D$, then $d^-(x_{i+2}, R) = 0$; (ii) if $y_t x_i \in D$, then $d^+(x_{i-2}, R) = 0$, where $i \in [1, k]$.

Proof. (i). Suppose, on the contrary, that for some $i \in [1, k]$ $x_i y_1 \in D$ and $d^-(x_{i+2}, R) \neq 0$. Without loss of generality, we assume that $x_i = x_1$. Then $d^-(x_3, R - \{y_1\}) = 0$ and $y_1 x_3 \in D$. It is easy to see that y_1, x_2 are non-adjacent and

$$d^-(x_2, \{y_1, y_2, \dots, y_{t-1}\}) = d^+(x_2, \{y_1, y_3, y_4, \dots, y_t\}) = 0, \quad \text{i.e.,} \quad d(x_2, R) \leq 2. \quad (6)$$

Since neither y_1 nor x_2 cannot be inserted into $C[x_3, x_1]$, using (2), (6) and Lemma 3.2, we obtain that

$$d(y_1) = d(y_1, C) + d(y_1, R) \leq k + n - k = n \quad \text{and} \quad d(x_2) = d(x_2, C) + d(x_2, R) \leq k + 2.$$

On the other hand, by Lemma 3.3 and (1) we have that $d^-(y_t) + d^+(y_1) \leq k + 2$ since the arc $y_t y_1$ cannot be inserted into C . Therefore, by condition A_0 , the following holds

$$3n - 2 \leq d(y_1) + d(x_2) + d^-(y_t) + d^+(y_1) \leq n + 2k + 4,$$

since y_1, x_2 are non-adjacent and $y_1 y_t \notin D$. From this and $k \leq n - 3$ it follows that $k = n - 3$, $x_2 y_2, y_2 y_1 \in D$ and hence, the cycle $x_2 y_2 y_1 x_3 x_4 \dots x_k x_1 x_2$ has length $k + 2$, which is a contradiction.

To show that (ii) is true, it is sufficient to apply the same arguments to the converse digraph of D . Claim 4 is proved. \square

Claim 5. If $t \geq 3$ and the vertices y_1, y_t are non-adjacent, then $t = 3$ and $y_3 y_2, y_2 y_1 \in D$.

Proof. Without loss of generality, we assume that $x_1 y_1, y_t x_2 \in D$ (since $\lambda = 1$).

Assume that $y_t y_i \in D$ for some $i \in [2, t - 2]$. Then $t \geq 4$. Since neither the arc $y_t y_i$ nor any vertex $y_j, j \in [1, t]$, cannot be inserted into C , using Lemma 3.1 and Lemma 3.3, we obtain that

$$d(y_j, C) \leq k \quad \text{and} \quad d^-(y_t, C) + d^+(y_i, C) \leq k. \quad (7)$$

From Claim 1 and the condition that y_1, y_t are non-adjacent it follows that

$$d(y_1, R) \leq n - k - 1 \quad \text{and} \quad d(y_t, R) \leq n - k - 1.$$

From this, since $d(y_j, C) \leq k$ for all $j \in [1, t]$ (by (7)), we obtain that $d(y_1)$ and $d(y_t) \leq n - 1$. Now using (1), (7) and apply condition A_0 to the triple of the vertices y_1, y_t, y_i , we obtain that

$$3n - 2 \leq d(y_1) + d(y_t) + d^-(y_t, C) + d^+(y_i, C) + d^-(y_t, R) + d^+(y_i, R) \leq 3n - 3,$$

which is a contradiction. Therefore, if $t \geq 4$, then $y_t y_i \notin D$ for all $i \in [2, t - 2]$.

In a similar way we can also show that $y_i y_1 \notin D$ for all $i \in [3, t - 1]$. Hence

$$d(y_1, R) \leq 2, \quad d(y_t, R) \leq 2 \quad \text{and} \quad d(y_1) + d(y_t) \leq 2k + 4, \quad (8)$$

since $d(y_i) \leq k$ for all $i \in [1, t]$.

If $t \geq 4$, then y_1, y_t and y_1, y_{t-1} are two distinct pairs of non-adjacent vertices. From (8) and $k \leq n - 4$ it follows that $d(y_1) + d(y_t) \leq 2n - 4$. On the other hand, since $d(y_1) \leq k + 2$, $d(y_{t-1}, C) \leq k - 1$ (by Claim 2 and Lemma 3.2(iii)) and $d(y_{t-1}, R) \leq n - k$ (by Claim 1), we have that

$$d(y_1) + d(y_{t-1}) \leq 2n - 3.$$

This together with $d(y_1) + d(y_t) \leq 2n - 4$ contradicts Lemma 3.5. Therefore $t = 3$.

Now we show that $y_3y_2 \in D$. Assume that this is false, i.e., $y_3y_2 \notin D$. Then we can apply condition A_0 to the triple of the vertices y_1, y_3, y_2 , since the vertices y_1, y_3 are non-adjacent and $y_3y_2 \notin D$. Notice that the arc y_2y_3 cannot be inserted into C and hence $d^-(y_2, C) + d^+(y_3, C) \leq k$ (by Lemma 3.3). Therefore by A_0 and Claim 2, we obtain that

$$3n - 2 \leq d(y_1) + d(y_3) + d^-(y_2) + d^+(y_3) \leq 3k + 4 \leq 3n - 5,$$

which is a contradiction. Therefore $y_3y_2 \in D$.

In a similar way, as above, we can show that $y_2y_1 \in D$. Claim 5 is proved. \square

Claim 6. If $t \geq 3$, then $y_t y_1 \in D$.

Proof. Suppose, on the contrary, that $t \geq 3$ and $y_t y_1 \notin D$, i.e., y_1, y_t are non-adjacent. Then by Claim 5, $t = 3$ and $y_3y_2, y_2y_1 \in D$. Without loss of generality, assume that x_1y_1 and $y_3x_2 \in D$ (since $\lambda = 1$). Notice that $d(y_1), d(y_3) \leq n - 1$ (by Lemma 3.1). We will distinguish two cases, according as there is an arc from R to $\{x_3, x_4, \dots, x_k\}$ or not.

Case 6.1. $A(R \rightarrow \{x_3, x_4, \dots, x_k\}) \neq \emptyset$. Then there exists a vertex x_l with $l \in [3, k]$ such that $d^-(x_l, R) \geq 1$ and $A(R \rightarrow \{x_3, x_4, \dots, x_{l-1}\}) = \emptyset$.

If $l = 3$, then from $d^-(x_3, \{y_2, y_3\}) = 0$ it follows that $y_1x_3 \in D$. From this it is easy to see that $d(x_2, \{y_1, y_2\}) = 0$. Since neither y_1 nor y_3 and nor x_2 cannot be inserted into $C[x_3, x_1]$ using Lemma 3.2 we obtain that $d(y_1), d(y_3)$ and $d(x_2) \leq n - 1$. Hence, $d(y_1) + d(y_3) \leq 2n - 2$ and $d(y_1) + d(x_2) \leq 2n - 2$, which contradicts Lemma 3.5.

Assume therefore that $l \geq 4$. From Claim 3, $x_1y_1 \in D$ and the minimality of l it follows that $d^+(x_{l-1}, R) \geq 1$. Without loss of generality, we may assume that $y_gx_l \in D$ and $x_{l-1}y_f$. It is easy to see that $y_f \neq y_g$, $y_f, y_g \in \{y_1, y_3\}$ and the vertices x_{l-1}, x_g are non-adjacent.

Assume first that $l = 4$. Then it is easy to see that $y_g = y_1$ and $y_f = y_3$, i.e., y_1x_4 and $x_3y_3 \in D$. Then clearly the vertices x_2, y_2 are non-adjacent and $x_2y_3 \notin D$. Therefore $x_2y_1 \notin D$ (for otherwise if $x_2y_1 \in D$, then Claim 3 is not true since $d^-(x_3, R) = 0$). Therefore $d(x_2, \{y_1, y_2\}) = 0$. Notice that x_2 cannot be inserted into the path $C[x_4, x_1]$ (for otherwise in D there is a cycle of length $n - 3$ for which Claim 5 is not true since $y_3x_3 \notin D$). Now by Lemma 3.2 and the above observation we obtain that

$$d(x_2) = d(x_2, C[x_4, x_1]) + d(x_2, R) + d(x_2, \{x_3\}) \leq n - 1.$$

Therefore $d(y_1) + d(x_2) \leq 2n - 2$, which together with $d(y_1) + d(y_3) \leq 2n - 2$ contradicts Lemma 3.5, since y_1, x_2 and y_1, y_3 are two distinct pairs of non-adjacent vertices.

Assume next that $l \geq 5$. From $x_1y_1 \in D$, $d^-(x_{l-1}, R) = 0$ and Claim 3 it follows that $A(\{x_2, x_3, \dots, x_{l-2}\} \rightarrow R) = \emptyset$, in particular, $x_2y_3 \notin D$. Therefore $A(\{x_3, x_4, \dots, x_{l-2}\}, R) = \emptyset$, $d(x_2, R) = 1$ (only $y_3x_2 \in D$), $d(x_{l-1}, R) = 1$ and

$$d(y_1, \{x_2, x_3, \dots, x_{l-2}\}) = d(y_3, \{x_3, x_4, \dots, x_{l-2}\}) = 0. \quad (9)$$

Since neither y_1 nor y_3 cannot be inserted into C , $x_2y_3 \notin D$ and $d^-(x_{l-1}, R) = 0$, using (9) and Lemma 3.2 we obtain that $d(y_1)$ and $d(y_3) \leq k - l + 5$. Therefore $d(y_1) + d(y_3) \leq 2n - 6$. Now we extend the path $P_0 := C[x_l, x_1]$ with the vertices x_2, x_3, \dots, x_{l-1} as much as possible. Then some vertices $z_1, z_2, \dots, z_d \in \{x_2, x_3, \dots, x_{l-1}\}$, $d \in [1, l - 2]$, are not on the extended path P_e . Therefore $d(z_i, C) \leq k + d - 1$ and hence, $d(z_i) \leq k + d$ for all $i \in [1, d]$. It is not difficult to show that there is a z_i which is not adjacent with y_1 . Thus we have $d(y_1) + d(z_i) \leq 2n - 3$. This together with $d(y_1) + d(y_3) \leq 2n - 6$ contradicts Lemma 3.5 since y_1, z_i and y_1, y_3 are two distinct pairs of non-adjacent vertices. In each case we have a contradiction, and hence the discussion of Case 6.1 is completed.

Case 6.2. $A(R \rightarrow \{x_3, x_4, \dots, x_k\}) = \emptyset$. Without loss of generality, we may assume that $A(\{x_3, x_4, \dots, x_k\} \rightarrow R) = \emptyset$ (for otherwise, we consider the converse digraph of D for which the considered Case

6.1 holds). Therefore $A(R, \{x_3, x_4, \dots, x_k\}) = \emptyset$. In particular, x_k is not adjacent with the vertices y_1 and y_3 . Notice that

$$d(y_1) = d(y_1, R) + d(y_1, C) \leq 2 + d(y_1, \{x_1, x_2\}) \leq 5,$$

$d(y_3) \leq 5$ and $d(x_k) = d(x_k, C) \leq 2n - 8$. Therefore $d(x_k) + d(y_1) \leq 2n - 3$ and $d(x_k) + d(y_3) \leq 2n - 3$, which contradicts Lemma 3.5. Claim 6 is proved. \square

Claim 7. If $t \geq 3$ and for some $i \in [1, k]$ $x_i y_1$, then $A(R \rightarrow C[x_{i+2}, x_{i-1}]) = \emptyset$.

Proof. Suppose that the claim is not true. Without loss of generality, we may assume that $x_1 y_1 \in D$ and $A(R \rightarrow \{x_3, x_4, \dots, x_k\}) \neq \emptyset$. Then there is a vertex x_l with $l \in [3, k]$ such that $d^-(x_l, R) \geq 1$ and if $l \geq 4$, then $A(R \rightarrow \{x_3, x_4, \dots, x_{l-1}\}) = \emptyset$. We have that $y_t y_1 \in D$ (by Claim 6). In particular, $y_t y_1 \in D$ implies that $\langle R \rangle$ is strong. On the other hand, by Claim 4(i), $d^-(x_3, R) = 0$ and hence, $l \geq 4$. From $x_1 y_1 \in D$ it follows that there exists a vertex x_r with $r \in [1, l-1]$ such that $d^+(x_r, R) \geq 1$. Choose r with these properties as maximal as possible. Let $x_r y_f$ and $y_g x_l \in D$. Notice that in $\langle R \rangle$ there is a (y_f, y_g) -path since $\langle R \rangle$ is strong. Using Claim 3 we obtain that $r = l - 1$. Then $y_f \neq y_g$ and in $\langle R \rangle$ any (y_f, y_g) -path is a Hamiltonian path. Since $\langle R \rangle$ is strong, from $d^-(x_{l-1}, R) = 0$, $d^-(x_l, R) \geq 1$ and from Claim 3 it follows that $A(\{x_2, x_3, \dots, x_{l-2}\} \rightarrow R) = \emptyset$, in particular, $d^+(x_2, R) = 0$. Then

$$A(\{x_3, x_4, \dots, x_{l-2}\}, R) = \emptyset, \quad d(y_1, \{x_2, x_3, \dots, x_{l-2}\}) = d(x_2, \{y_1, y_2, \dots, y_{t-1}\}) = 0. \quad (10)$$

Note that x_2, y_1 and x_2, y_2 are two distinct pairs of non-adjacent vertices. We extend the path $P_0 := C[x_l, x_1]$ with the vertices x_2, x_3, \dots, x_{l-1} as much as possible. Then some vertices $z_1, z_2, \dots, z_d \in \{x_2, x_3, \dots, x_{l-1}\}$, where $d \in [1, l-2]$, are not on the extended path P_e (for otherwise, since in $\langle R \rangle$ there is a (y_f, y_g) -path, using the path P_{e-1} or P_e we obtain a non-Hamiltonian cycle longer than C). By Lemma 3.2, for all $i \in [1, d]$ we have that

$$d(z_i, C) \leq k + d - 1 \quad \text{and} \quad d(z_i) = d(z_i, C) + d(z_i, R) \leq k + d - 1 + d(z_i, R). \quad (11)$$

Assume that there is a vertex $z_i \neq x_{l-1}$. Then, by (10), $d(z_i, R) \leq 1$ (since $d(x_2, R) \leq 1$). Notice that y_1, z_i and y_2, z_i are two distinct pairs of non-adjacent vertices (by (10)). Since neither y_1 nor y_2 cannot be inserted into $C[x_{l-1}, x_1]$ and $y_1 x_{l-1} \notin D$, $y_2 x_{l-1} \notin D$, by Lemma 3.2(ii) and (10) for $j = 1$ and 2 we obtain that

$$d(y_j, C) = d(y_j, C[x_{l-1}, x_1]) \leq k - l + 3. \quad (12)$$

In particular, by (2),

$$d(y_1) = d(y_1, C) + d(y_1, R) \leq k - l + 3 + n - k = n - l + 3.$$

This together with (11) and $d(z_i, R) \leq 1$ implies that

$$d(y_1) + d(z_i) \leq 2n - 2,$$

since $k \leq n - 3$ and $d \leq l - 2$. Therefore, by Lemma 3.5, $d(y_2) + d(z_i) \geq 2n - 1$. Hence, by (2) and (11) we have

$$2n - 1 \leq d(y_2) + d(z_i) \leq n + d + d(z_i, R) + d(y_2, C).$$

From this and (12) it follows that $d(y_2, C) = k - l + 3$, $d(z_i, R) = 1$ and $k = n - 3$. Then $z_i = x_2$ and $y_t x_2 \in D$ (by (10) and $d^+(x_2, R) = 0$). Therefore $x_1 y_2 \notin D$. From this, $y_2 x_{l-1} \notin D$ and $d(y_2, C) = k - l + 3$, by Lemma 3.2(iii) we conclude that y_2 can be inserted into C , which contrary to our assumption.

Now assume that there is no $z_i \neq x_{l-1}$. Then $d = 1$, $z_1 = x_{l-1}$ and $d^-(x_l, \{y_2, y_3, \dots, y_t\}) = 0$ (since $x_1 y_1 \in D$). Therefore $y_1 x_l \in D$ and hence, $d(x_{l-1}, R - \{y_2\}) = 0$ (since $y_t y_1 \in D$ and l is minimal), in particular, the vertices y_t, x_{l-1} are non-adjacent. This together with (11) implies that $d(x_{l-1}) \leq k + 1$

(only $x_{l-1}y_2 \in D$ is possible). Notice that neither y_t nor the arc $y_t y_1$ cannot be inserted into C , and therefore, by Lemmas 3.2, 3.3 and by (1), (2) we obtain that $d(y_t) \leq n$ and $d^-(y_t) + d^+(y_1) \leq k + 2$. Now for the triple of the vertices y_t, x_{l-1}, y_1 , by condition A_0 , we obtain that

$$3n - 2 \leq d(x_{l-1}) + d(y_t) + d^-(y_t) + d^+(y_1) \leq 3n - 3$$

since $k \leq n - 3$, which is a contradiction. Claim 7 is proved. \square

Claim 8. If $t \geq 3$, $x_1 y_1$ and $y_t x_2 \in D$, then $d^-(x_1, R) = 0$.

Proof. Assume that $d^-(x_1, R) \geq 1$. By Claim 6, $y_t y_1 \in D$. Now using Claims 4(ii) and 7, we obtain that $d^+(x_k, R) = 0$ and

$$A(R \rightarrow \{x_3, x_4, \dots, x_k\}) = \emptyset. \quad (13)$$

In particular, $d(x_k, R) = 0$. This together with $d^-(x_1, R) \geq 1$, (13) and Claim 3 implies that $A(\{x_2, x_3, \dots, x_{k-1}\} \rightarrow R) = \emptyset$. Now again using (13) we get that $A(\{x_3, x_4, \dots, x_k\}, R) = \emptyset$. This together with $d^+(x_2, R) = d^-(x_2, \{y_1, y_2, \dots, y_{t-1}\}) = 0$ implies that $d(x_2, R) = 1$, $d(y_2, C) \leq 1$ (only $y_2 x_1 \in D$ is possible) and $d(x_3, R) = 0$. Therefore, by (2),

$$d(y_2) + d(x_3) = d(y_2, C) + d(y_2, R) + d(x_3, R) + d(x_3, C) \leq n + k \leq 2n - 3$$

and $d(y_2) + d(x_2) \leq 2n - 2$, which contradicts Lemma 3.5 since y_2, x_3 and y_2, x_2 are two distinct pairs of non-adjacent vertices. This completes the proof of Claim 8. \square

Claim 9. If $t \geq 3$, $x_1 y_1$ and $y_t x_2 \in D$, then $A(\{x_3, x_4, \dots, x_k\} \rightarrow R) = \emptyset$.

Proof. By Claim 6, $y_t y_1 \in D$. Suppose that $A(\{x_3, x_4, \dots, x_k\} \rightarrow R) \neq \emptyset$. Recall that Claim 4(ii) implies that $d^+(x_k, R) = 0$. Let x_r , $r \in [3, k - 1]$, be chosen so that $x_r y_i \in D$ for some $i \in [1, t]$ and r is maximum possible. Then $A(\{x_{r+1}, x_{r+2}, \dots, x_k\}, R) = \emptyset$ and $d^-(x_1, R) = 0$ by Claims 7 and 8, respectively. This together with $y_t x_2 \in D$ contradicts Claim 3. Claim 9 is proved. \square

We are now ready to complete the proof of Theorem 1.10 for Part 1 (when $k \leq n - 3$, i.e., $t \geq 3$). By Claim 6, if $t \geq 3$, then $y_t y_1 \in D$. Without loss of generality, we may assume that $x_1 y_1$ and $y_t x_2 \in D$ since $\lambda = 1$. Then from Claims 7, 8 and 9 it follows that

$$A(R \rightarrow \{x_3, x_4, \dots, x_k, x_1\}) = A(\{x_3, x_4, \dots, x_k\} \rightarrow R) = \emptyset.$$

From this and

$$d^-(x_2, \{y_1, y_2, \dots, y_{t-1}\}) = d^+(x_1, \{y_2, y_3, \dots, y_t\}) = 0$$

we obtain that x_1, y_2 and x_1, y_t are two distinct pairs of non-adjacent vertices and $d(y_2, C) \leq 1$, $d(y_t, C) \leq 2$, $d(x_1, R) = 1$. Therefore $d(y_2) \leq n - k + 2$, $d(y_t) \leq n - k + 2$ (by (2)) and $d(x_1) \leq 2k - 1$. These inequalities imply that $d(y_2) + d(x_1) \leq 2n - 2$ and $d(y_t) + d(x_1) \leq 2n - 2$, which contradicts Lemma 3.5. and completes the discussion of Part 1.

Part 2. $k = n - 2$, i.e., $t = 2$. For this part first we will prove Claims 10-15 below.

Claim 10. If $x_i y_f \in D$ and $y_2 y_1 \notin D$, where $i \in [1, n - 2]$ and $f \in [1, 2]$, then there is no $l \in [3, n - 2]$ such that $y_f x_{i+l-1} \in D$ and $d(y_f, \{x_{i+1}, x_{i+2}, \dots, x_{i+l-2}\}) = 0$.

Proof. The proof is by contradiction. Suppose that $x_i y_f, y_f x_{i+l-1} \in D$ and $d(y_f, \{x_{i+1}, x_{i+2}, \dots, x_{i+l-2}\}) = 0$ for some $l \in [3, n - 2]$. Without loss of generality, we may assume that $x_i = x_1$. Then $x_1 y_f, y_f x_l \in D$ and $d(y_f, \{x_2, x_3, \dots, x_{l-1}\}) = 0$. Since D contains no cycle of length $n - 1$, using Lemmas 3.2 and 3.3, we obtain that

$$d^-(y_1) + d^+(y_2) \leq n - 2 \quad \text{and} \quad d(y_f) \leq n - l + 2. \quad (14)$$

We extend the path $P_0 := C[x_l, x_1]$ with the vertices x_2, x_3, \dots, x_{l-1} as much as possible. Then some vertices $z_1, z_2, \dots, z_d \in \{x_2, x_3, \dots, x_{l-1}\}$, $d \in [1, l-2]$, are not on the extended path P_e . Therefore by Lemma 3.2, $d(z_1) = d(z_1, C) + d(z_1, \{y_{3-f}\}) \leq n + d - 1$. Now, since the vertices y_f, z_1 are non-adjacent and $y_2 y_1 \notin D$, by condition A_0 and (14) we have

$$3n - 2 \leq d(y_f) + d(z_1) + d^-(y_1) + d^+(y_2) \leq 3n - 3,$$

a contradiction. Claim 10 is proved. \square

Claim 11. $y_2 y_1 \in D$ (i.e., if $k = n - 2$, then $\langle V(D) - V(C) \rangle$ is strong).

Proof. Suppose, on the contrary, that $y_2 y_1 \notin D$. Without loss of generality, we may assume that $x_1 y_1 \in D$ and the vertices y_1, x_2 are non-adjacent. Then $y_2 x_3 \notin D$ and since D contains no cycle of length $n - 1$, using Lemma 3.3 for the arc $y_1 y_2$ we obtain that

$$d^-(y_1) + d^+(y_2) \leq n - 2. \quad (15)$$

Case 11.1. $d^+(y_1, C[x_3, x_{n-2}]) \geq 1$. Let x_l , $l \in [3, n-2]$, be chosen so that $y_1 x_l \in D$ and l is minimum, i.e., $d^+(y_1, C[x_2, x_{l-1}]) = 0$. It is easy to see that the vertices y_1 and x_{l-1} are non-adjacent. By Claim 10 we can assume that $l \geq 5$ (if $l \leq 4$, then $d(y_1, C[x_2, x_{l-1}]) = 0$, a contradiction to Claim 10) and $d^-(y_1, C[x_3, x_{l-2}]) \geq 1$. It follows that there exists a vertex x_r with $r \in [3, l-2]$ such that $x_r y_1 \in D$ and $d(y_1, C[x_{r+1}, x_{l-1}]) = 0$. Consequently, for the vertices y_1 , x_r and x_l Claim 10 is not true, a contradiction.

Case 11.2. $d^+(y_1, C[x_3, x_{n-2}]) = 0$. Then $d^+(y_1, C[x_2, x_{n-2}]) = 0$ and either $y_1 x_1 \in D$ or $y_1 x_1 \notin D$.

Subcase 11.2.1. $y_1 x_1 \in D$. Then $x_{n-2} y_1 \notin D$ and hence, the vertices y_1, x_{n-2} are non-adjacent. Claim 10 implies that $d^-(y_1, C[x_2, x_{n-2}]) = 0$. This together with $d^+(y_1, C[x_2, x_{n-2}]) = 0$ and $y_2 y_1 \notin D$ gives $d(y_1) = 3$. Clearly, $d(x_2) \leq 2n - 4$ and hence, for the vertices y_1, y_2, x_2 by condition A_0 and (15) we have,

$$3n - 2 \leq d(y_1) + d(x_2) + d^-(y_1) + d^+(y_2) \leq 3n - 3,$$

which is a contradiction.

Subcase 11.2.2. $y_1 x_1 \notin D$. Then $d^+(y_1, C) = 0$, $d^+(y_1) = 1$ and $d^+(y_2, C) \geq 1$ since D is strong. Without loss of generality, we may assume that $d^-(y_2, C) = 0$ (for otherwise for the vertex y_2 in the converse digraph of D we would have the above considered Case 11.1 or Subcase 11.2.1). Using Lemma 3.5, it is not difficult to show that $n \geq 6$.

Suppose first that $y_2 x_2 \in D$. Then $x_{n-2} y_1 \notin D$ and hence, the vertices x_{n-2}, y_1 are non-adjacent.

Let for some $l \in [3, n-3]$ $x_l y_1 \in D$ and $d^-(y_1, C[x_{l+1}, x_{n-2}]) = 0$. Then $d(y_1, C[x_{l+1}, x_{n-2}]) = 0$ and $d(y_1) \leq l$ since $d^+(y_1, C) = 0$ and x_2, y_1 are non-adjacent. Extend the path $P_0 := C[x_2, x_l]$ with the vertices $x_{l+1}, x_{l+2}, \dots, x_{n-2}, x_1$ as much as possible. Then some vertices $z_1, z_2, \dots, z_d \in \{x_{l+1}, x_{l+2}, \dots, x_{n-2}, x_1\}$, $d \in [2, n-l-1]$, are not on the extended path P_e . For a vertex $z_i \neq x_1$ by Lemma 3.2 we obtain that $d(z_i) = d(z_i, C) + d(z_i, \{y_2\}) \leq n + d - 1$. Therefore, since $y_2 y_1 \notin D$ and the vertices z_i, y_1 are non-adjacent, by condition A_0 and (15), we get that

$$3n - 2 \leq d(y_1) + d(z_i) + d^-(y_1) + d^+(y_2) \leq 3n - 4,$$

which is a contradiction.

Let now $x_l y_1 \notin D$ for all $l \in [3, n-2]$, i.e., $d^-(y_1, C[x_3, x_{n-2}]) = 0$. Then from $d^+(y_1, C[x_2, x_{n-2}]) = 0$ and $x_{n-2} y_2 \notin D$ (since $d^-(y_2, C) = 0$) it follows that $d(y_1) = 2$ and $d(x_{n-2}) \leq 2n - 5$. From this, since the vertices y_1, x_{n-2} are non-adjacent and $y_2 y_1 \notin D$, by condition A_0 and (15) we have that

$$3n - 2 \leq d(y_1) + d(x_{n-2}) + d^-(y_1) + d^+(y_2) \leq 3n - 5,$$

which is a contradiction.

Suppose next that $y_2x_2 \notin D$. Then $d(y_2, \{x_2, x_3\}) = 0$, since $d^-(y_2, C) = 0$. Let for some $l \in [4, n-2]$ $y_2x_l \in D$ and $d^+(y_2, C[x_2, x_{l-1}]) = 0$. Then $d(y_2, C[x_2, x_{l-1}]) = 0$ and the vertices y_1, x_{l-2} are non-adjacent since $d^+(y_1, C[x_2, x_{n-2}]) = 0$. It is easy to see that there exists a vertex $x_r \in \{x_1, x_2, \dots, x_{l-3}\}$ such that $x_r y_1 \in D$ and $d(y_1, C[x_{r+1}, x_{l-2}]) = 0$. Thus we have that $A(R, C[x_{r+1}, x_{l-2}]) = \emptyset$. Notice that $d(y_2) \leq n - l + 1$ since $d^-(y_2, C) = 0$ and $d(y_2, C[x_2, x_{l-1}]) = 0$. We extend the path $P_0 := C[x_l, x_r]$ with the vertices $x_{r+1}, x_{r+2}, \dots, x_{l-1}$ as much as possible. Then some vertices $z_1, z_2, \dots, z_d \in \{x_{r+1}, x_{r+2}, \dots, x_{l-1}\}$, $d \in [2, l - r - 1]$, are not on the extended path P_e . Therefore by Lemma 3.2 for $z_i \neq x_{l-1}$ we have, $d(z_i) \leq n + d - 3$. Now by condition A_0 and (15) we obtain that

$$3n - 2 \leq d(y_2) + d(z_i) + d^-(y_1) + d^+(y_2) < 3n - 3,$$

a contradiction. Let now $d^+(y_2, \{x_2, x_3, \dots, x_{n-2}\}) = 0$. Then $d(y_2) = 2$, $d(x_2) \leq 2n - 6$ and the vertices x_2, y_2 are non-adjacent. By condition A_0 we have

$$3n - 2 \leq d(y_2) + d(x_2) + d^-(y_1) + d^+(y_2) < 3n - 3,$$

a contradiction. Claim 11 is proved. \square

Claim 12. For any $i \in [1, n-2]$ and $f \in [1, 2]$ the following holds

i) $d^-(y_f, \{x_{i-1}, x_i\}) \leq 1$ and ii) $d^+(y_f, \{x_{i-1}, x_i\}) \leq 1$.

Proof. Suppose that the claim is not true. Without loss of generality, we may assume that $x_{n-3}y_1, x_{n-2}y_1 \in D$ and y_1, x_1 are non-adjacent. By Claim 11, $y_2y_1 \in D$. It is easy to see that $d^+(y_2, \{x_1, x_2\}) = 0$, $y_1x_{n-2} \notin D$ and $y_1x_2 \notin D$ (for otherwise, if $y_1x_2 \in D$, then $x_{n-2}y_1x_2x_3 \dots x_{n-3}x_{n-2}$ is a cycle of length $n-2$ for which $\langle \{y_2, x_1\} \rangle$ is not strong, a contradiction to Claim 11). Therefore, $A(R \rightarrow \{x_1, x_2\}) = \emptyset$. It is not difficult to check that $n \geq 6$.

Assume first that $A(R \rightarrow \{x_3, x_4, \dots, x_{n-3}\}) \neq \emptyset$. Now let x_l , $l \in [3, n-3]$, be the first vertex after x_2 that $d^-(x_l, R) \geq 1$. Then $A(R \rightarrow \{x_1, x_2, \dots, x_{l-1}\}) = \emptyset$ since $A(R \rightarrow \{x_1, x_2\}) = \emptyset$ (in particular, $d^-(x_{l-1}, R) = \emptyset$). From the minimality of l and $x_{n-2}y_1 \in D$ it follows that there is a vertex $x_r \in \{x_{n-2}, x_1, x_2, \dots, x_{l-2}\}$ such that $d^+(x_r, R) \geq 1$ and $A(\{x_{r+1}, x_{r+2}, \dots, x_{l-2}\}, R) = \emptyset$ (if $x_r = x_{n-2}$, then $x_{r+1} = x_1$). This is contrary to Claim 3 since $d^-(x_{l-1}, R) = 0$ and $\langle R \rangle$ is strong.

Assume next that $A(R \rightarrow \{x_3, x_4, \dots, x_{n-3}\}) = \emptyset$. This together with $A(R \rightarrow \{x_1, x_2\}) = \emptyset$ gives that $A(R \rightarrow \{x_1, x_2, \dots, x_{n-3}\}) = \emptyset$. From this, since D is strong and $y_1x_{n-2} \notin D$, it follows that $y_2x_{n-2} \in D$. Then $x_{n-3}y_2 \notin D$ and $x_{n-4}y_1 \notin D$. Now using Claim 11 we obtain that $d(y_2, \{x_{n-4}, x_{n-3}\}) = 0$ and $d(x_{n-4}, R) = 0$. If $A(\{x_1, x_2, \dots, x_{n-5}\} \rightarrow R) \neq \emptyset$, then there is a vertex x_r with $r \in [1, n-5]$ such that $d^+(x_r, R) \geq 1$ and $A(R, \{x_{r+1}, x_{r+2}, \dots, x_{n-4}\}) = \emptyset$ ($n \geq 6$) which contradicts Claim 3, since $y_2x_{n-2} \in D$ and $d^-(x_{n-3}, R) = 0$. Assume therefore that $A(\{x_1, x_2, \dots, x_{n-4}\} \rightarrow R) = \emptyset$. Thus we have that $A(\{x_1, x_2, \dots, x_{n-4}\}, R) = \emptyset$ and $d^-(x_{n-3}, R) = 0$. Then $d(y_1) = 4$, $d(y_2) \leq 4$ and $d(x_1) \leq 2n - 6$. From this it follows that $d(y_1) + d(x_1) \leq 2n - 2$ and $d(y_2) + d(x_1) \leq 2n - 2$ which contradicts Lemma 3.5. This contradiction proves that $d^-(y_f, \{x_{i-1}, x_i\}) \leq 1$ for all $i \in [1, n-2]$ and $f \in [1, 2]$. Similarly, one can show that $d^+(y_f, \{x_{i-1}, x_i\}) \leq 1$. Claim 12 is proved. \square

Claim 13. If $x_i y_f \in D$ (respectively, $y_f x_i \in D$), then $d(y_f, \{x_{i+2}\}) \neq 0$ (respectively, $d(y_f, \{x_{i-2}\}) \neq 0$), where $i \in [1, n-2]$ and $f \in [1, 2]$.

Proof. Suppose that the claim is not true. By Claim 11, $y_2y_1 \in D$. Without loss of generality, we may assume that $x_{n-2}y_1 \in D$ and $d(y_1, \{x_2\}) = 0$, i.e., the vertices y_1 and x_2 are non-adjacent. Claim 12 implies that the vertices y_1, x_1 also are non-adjacent. Note that $y_2x_2 \notin D$ and hence $d^-(x_2, R) = 0$. Now it is not difficult to see that if $n = 5$, then $d(y_1) + d(x_1) \leq 8$ and $d(y_1) + d(x_2) \leq 8$, a contradiction

to Lemma 3.5. Assume therefore that $n \geq 6$ and consider the following cases.

Case 13.1. $A(R \rightarrow \{x_3, x_4, \dots, x_{n-3}\}) \neq \emptyset$. Then there is a vertex x_l with $l \in [3, n-3]$ such that $d^-(x_l, R) \geq 1$ and $A(R \rightarrow \{x_2, x_3, \dots, x_{l-1}\}) = \emptyset$. We now consider the case $l = 3$ and the case $l \geq 4$ separately.

Assume that $l = 3$. Then $y_2x_3 \in D$ or $y_1x_3 \in D$.

Let $y_2x_3 \in D$. Then the vertices y_2, x_2 are non-adjacent. Since the vertices y_1, x_2 are non-adjacent Claim 11 implies that $x_1y_2 \notin D$. This contradicts Claim 3 because of $d(x_2, R) = 0$ and $d^+(x_1, R) = 0$.

Let now $y_1x_3 \in D$ and $y_2x_3 \notin D$. Then it is easy to see that $x_1y_2 \notin D$ and $y_2x_2 \notin D$. From this and Claim 11 implies that neither x_1 nor x_2 cannot be inserted into $C[x_3, x_{n-2}]$. Notice that if $x_2y_2 \in D$, then $x_{n-2}x_2 \notin D$, and if $y_2x_1 \in D$, then $x_1x_3 \notin D$. Now using Lemma 3.2, we obtain that $d(y_1), d(x_1)$ and $d(x_2) \leq n-1$ since $d(y_1, \{x_1, x_2\}) = 0$. Therefore $d(y_1) + d(x_1) \leq 2n-2$ and $d(y_1) + d(x_2) \leq 2n-2$, which contradicts Lemma 3.5 since y_1, x_1 and y_1, x_2 are two distinct pairs of non-adjacent vertices. This contradiction completes the discussion of Case 13.1 when $l = 3$.

Assume that $l \geq 4$. Let $y_gx_l \in D$, where $g \in [1, 2]$. Then, by the minimality of l , the vertices y_g, x_{l-1} are non-adjacent, $y_{3-g}x_{l-1} \notin D$ and $x_{l-2}y_{3-g} \notin D$. Hence by Claim 11 we get that $x_{l-2}y_g \notin D$. From the minimality of l and $d^-(x_2, R) = 0$ (for $l = 4$) it follows that x_{l-2} is not adjacent with y_1 and y_2 , i.e., $d(x_{l-2}, R) = 0$. This together with $d^-(x_2, R) = d^-(x_{l-1}, R) = 0$ and Claim 3 implies that $A(R, \{x_2, x_3, \dots, x_{l-2}\}) = \emptyset$, $d^+(x_1, R) = 0$, $d^-(x_1, R) \geq 1$ and $d^+(x_{l-1}, R) \geq 1$. It follows that $y_2x_1 \in D$ since $y_1x_1 \notin D$.

Assume first that $y_g = y_2$. Then $x_{l-1}y_1 \in D$. Using Lemma 3.2(ii) and

$$d(y_1, C[x_1, x_{l-2}]) = d(y_2, C[x_2, x_{l-1}]) = 0$$

we obtain that

$$\begin{aligned} d(y_1) &= d(y_1, \{y_2\}) + d(y_1, C[x_{l-1}, x_{n-2}]) \leq n - l + 2 \quad \text{and} \\ d(y_2) &= d(y_2, \{y_1\}) + d(y_2, C[x_l, x_1]) \leq n - l + 2. \end{aligned} \tag{16}$$

Now we extend the path $P_0 := C[x_l, x_{n-2}]$ with the vertices x_1, x_2, \dots, x_{l-1} as much as possible. Then some vertices $z_1, z_2, \dots, z_d \in \{x_1, x_2, \dots, x_{l-1}\}$, $d \in [2, l-1]$, are not on the extended path P_e . Therefore by Lemma 3.2, we have that $d(z_i, C) \leq n + d - 3$. If there is a $z_i \notin \{x_1, x_{l-1}\}$, then $d(z_i) \leq n + d - 3$ and by (16), $d(z_i) + d(y_1) \leq 2n - 2$ and $d(z_i) + d(y_2) \leq 2n - 2$, which contradicts Lemma 3.5 since z_i is not adjacent with y_1 and y_2 . Therefore assume that $\{z_1, z_2\} = \{x_1, x_{l-1}\}$ ($d = 2$). Then P_e ($e = l - 3 \geq 1$) is an (x_l, x_{n-2}) -path with vertex set $V(C) - \{x_1, x_{l-1}\}$. Thus, we have that $y_2P_e y_1 y_2$ is a cycle of length $n - 2$. Therefore, by Claim 11, $x_1x_{l-1} \in D$, and hence $x_1x_{l-1}P_{e-1}y_1y_2x_1$ is a cycle of length $n - 1$, which is a contradiction to our supposition.

Assume second that $y_g = y_1$. Then $x_{l-1}y_2 \in D$ and $d(y_1, C[x_1, x_{l-1}]) = 0$. Using Lemma 3.2, we obtain that for this case (16) also holds, since $x_1y_2 \notin D$ and $y_2x_{l-1} \notin D$. Again we extend the path $C[x_l, x_{n-2}]$ with vertices x_1, x_2, \dots, x_{l-1} as much as possible. Then some vertices $z_1, z_2, \dots, z_d \in \{x_1, x_2, \dots, x_{l-1}\}$, $d \in [1, l-1]$, are not on the extended path P_e . Similarly to the first case, we obtain that $z_i \notin \{x_2, x_3, \dots, x_{l-2}\}$ (i.e., $z_i = x_1$ or $z_i = x_{l-1}$) and $d(z_i) \leq n + d - 2$. Notice that $C' := y_1P_e y_1$ is a cycle of length $n - d - 1$ with vertex set $V(C) \cup \{y_1\} - \{z_1, z_d\}$. From Claim 11 it follows that $d = 2$, i.e., $\{z_1, z_d\} = \{x_1, x_{l-1}\}$. From (16) and $d(z_i) \leq n + d - 2$ we obtain that $d(y_1) + d(x_1) \leq 2n - 2$ and $d(y_1) + d(x_{l-1}) \leq 2n - 2$, which contradicts Lemma 3.5, since y_1, x_1 and y_1, x_{l-1} are two distinct pairs of non-adjacent vertices.

Case 13.2. $A(R \rightarrow \{x_3, x_4, \dots, x_{n-3}\}) = \emptyset$. Then $A(R \rightarrow \{x_{n-2}, x_1\}) \neq \emptyset$ since $d^-(x_2, R) = 0$ and D is strong, and y_1, x_{n-3} are non-adjacent (by Claim 12). For this case we distinguish three subcases.

Subcase 13.2.1. $y_2x_{n-2} \in D$. Then it is easy to see that $d(x_{n-3}, R) = 0$. This together with $y_2x_{n-2} \in D$ and Claim 3 implies that $A(\{x_1, x_2, \dots, x_{n-3}\} \rightarrow R) = \emptyset$. Therefore $d(R, \{x_2, x_3, \dots, x_{n-3}\}) = \emptyset$ and $d(y_1), d(y_2) \leq 4$ (since $y_2x_1 \notin D$ by Claim 12) and $d(x_{n-3}) \leq 2n - 6$. From these it follows that $d(y_1) + d(x_{n-3}) \leq 2n - 2$ and $d(y_2) + d(x_{n-3}) \leq 2n - 2$, which contradicts Lemma 3.5.

Subcase 13.2.2. $y_2x_{n-2} \notin D$ and $y_2x_1 \in D$. Then using Claim 12 it is easy to see that y_2 and x_{n-2} are non-adjacent.

Let $x_{n-3}y_2 \in D$. Then $y_1x_{n-2} \in D$ (by Claim 11). Using Claims 11 and 12 we obtain that x_{n-4} is not adjacent with y_1 and y_2 . From Claim 3 it follows that $A(\{x_1, x_2, \dots, x_{n-4}\} \rightarrow R) = \emptyset$ and $A(R, C[x_2, x_{n-4}]) = \emptyset$. Therefore $d(y_1) = d(y_2) = 4$ and $d(x_2) \leq 2n - 6$. From these it follows that $d(y_1) + d(x_2) \leq 2n - 2$ and $d(y_2) + d(x_2) \leq 2n - 2$, which contradicts Lemma 3.5 since x_2, y_1 and x_2, y_2 are two distinct pairs of non-adjacent vertices.

Let now $x_{n-3}y_2 \notin D$. Then y_2, x_{n-3} are non-adjacent and hence, $d(x_{n-3}, R) = 0$. Now from Claim 3 it follows that $A(\{x_2, x_3, \dots, x_{n-3}\} \rightarrow R) = \emptyset$. Therefore

$$d(y_1, C[x_1, x_{n-3}]) = d(y_2, C[x_2, x_{n-2}]) = 0,$$

$d(y_1) \leq 4$, $d(y_2) \leq 4$ and $d(x_2) \leq 2n - 6$. This contradicts Lemma 3.5 since x_2, y_1 and x_2, y_2 are two distinct pairs of non-adjacent vertices.

Subcase 13.2.3. $y_2x_{n-2} \notin D$ and $y_2x_1 \notin D$. Then $y_1x_{n-2} \in D$ (since D is strong), the vertex y_1 is not adjacent with vertices x_{n-3}, x_{n-4} and $x_{n-4}y_2 \notin D$, i.e., the vertices y_2, x_{n-4} also are non-adjacent. Using Claim 3, we can assume that $A(C[x_1, x_{n-4}] \rightarrow R) = \emptyset$. Therefore $d(y_1) = 4$, $d(y_2) \leq 3$ and $d(x_1) \leq 2n - 6$. This contradicts Lemma 3.5 since x_1 is not adjacent with y_1 and y_2 . This completes the proof of Claim 13. \square

Claim 14. If $x_iy_f \in D$ and the vertices y_f, x_{i+1} are non-adjacent, then the vertices x_{i+1}, y_{3-f} are adjacent, where $i \in [1, n - 2]$ and $f \in [1, 2]$.

Proof. Without loss of generality, we may assume that $x_i = x_{n-2}$ (i.e., $x_{i+1} = x_1$) and $y_f = y_1$. Suppose, on the contrary, that x_1, y_2 are non-adjacent. From Claims 11 and 13 it follows that $y_1x_2 \notin D$ and $x_2y_1 \in D$. Therefore $A(R \rightarrow \{x_1, x_2\}) = \emptyset$. If $n = 5$, then $x_2y_1, x_3y_1 \in D$ which contradicts Claim 12. Assume therefore that $n \geq 6$. As D is strong there is a vertex x_l with $l \in [3, n - 2]$ such that $d^-(x_l, R) \geq 1$ (say $y_gx_l \in D$) and $A(R \rightarrow C[x_1, x_{l-1}]) = \emptyset$. Then the vertices x_{l-1}, y_g are non-adjacent and $d(x_{l-2}, R) = 0$ (by $x_{l-2}y_{3-g} \notin D$ and by Claim 11). Now, since $x_{n-2}y_1$ and $x_2y_1 \in D$, there exists a vertex $x_r \in C[x_{n-2}, x_{l-3}]$ (if $l = 3$, then $x_{n-2} = x_{l-3}$) such that $d^+(x_r, R) \geq 1$ and $A(R, C[x_{r+1}, x_{l-2}]) = \emptyset$. This contradicts Claim 3. Claim 14 is proved. \square

Claim 15. If $x_iy_j \in D$, where $i \in [1, n - 2]$ and $j \in [1, 2]$, then $y_jx_{i+2} \in D$.

Proof. Without loss of generality, we may assume that $x_i = x_{n-2}$ and $y_j = y_1$. Suppose that the claim is not true, that is $x_{n-2}y_1 \in D$ and $y_1x_2 \notin D$. Then, by Claims 12 and 13, the vertices y_1, x_1 are non-adjacent, $x_2y_1 \in D$ (hence, $n \geq 6$) and y_1, x_3 are also non-adjacent. From this, by Claim 14 we obtain that the vertex y_2 is adjacent with vertices x_1 and x_3 . Therefore either $y_2x_3 \in D$ or $x_3y_2 \in D$.

Case 15.1. $y_2x_3 \in D$. Then x_2, y_2 are non-adjacent (by Claim 12), $x_2x_1 \in D$ and $x_1y_2 \notin D$ by Claim 11 (for otherwise D would have a cycle C' of length $n - 2$ for which $\langle V(D) - V(C') \rangle$ is not strong). Notice that $y_2x_1 \in D$. Since neither y_1 nor y_2 cannot be inserted into C , $y_1x_2 \notin D$ and y_1, x_1 are non-adjacent (respectively, $x_1y_2 \notin D$ and y_2, x_2 are non-adjacent) using Lemma 3.2(ii), we obtain that

$$d(y_1) \leq n - 1 \quad \text{and} \quad d(y_2) \leq n - 1. \quad (17)$$

Notice that $x_{n-2}x_2 \notin D$ and $x_1x_3 \notin D$. Therefore, since neither x_1 nor x_2 cannot be inserted into

$C[x_3, x_{n-2}]$ (otherwise we obtain a cycle of length $n - 1$), again using Lemma 3.2(ii), we obtain that

$$d(x_1) \leq n - 1 \quad \text{and} \quad d(x_2) \leq n - 1. \quad (18)$$

It is easy to check that $n \geq 7$.

Remark. Observe that from (17), (18) and Lemma 3.5 it follows that if $x_i \neq x_1$ and y_1, x_i are non-adjacent or $x_i \neq x_2$ and x_i, y_2 are non-adjacent, then $d(x_i) \geq n$.

Assume first that $d^+(y_1, C[x_4, x_{n-2}]) \geq 1$. Let $x_l, l \in [4, n - 2]$, be the first vertex after x_3 that $y_1 x_l \in D$. Then the vertices y_1 and x_{l-1} are non-adjacent. Therefore y_1 and x_{l-2} are adjacent (by Claim 13) and hence, $x_{l-2} y_1 \in D$ because of $x_2 y_1 \in D$ and minimality of l ($l - 1 \neq 4$). Since x_{l-1} cannot be inserted into $C[x_l, x_{l-2}]$, using Lemma 3.2 and the above Remark, we obtain that $d(x_{l-1}) = n$ and hence, $d(y_1) = n - 1$ (by Lemma 3.5). This together with $d(y_1, \{x_1, x_2, x_3, y_2\}) = 3$ implies that $d(y_1, C[x_4, x_{n-2}]) = n - 4$. Again using Lemma 3.2, we obtain that $y_1 x_4 \in D$ (since $|C[x_4, x_{n-2}]| = n - 5$). Thus $y_1 C[x_4, x_2] y_1$ is a cycle of length $n - 2$. Therefore, $x_3 y_2 \in D$ (by Claim 11), $y_1 x_5 \notin D$ and the vertices y_2, x_4 are non-adjacent (by Claim 12). From $y_1 x_5 \notin D$ (by Lemma 3.2) we obtain that $d(y_1, C[x_5, x_{n-2}]) \leq n - 6$. Therefore $x_4 y_1 \in D$ and $d(y_1, C[x_5, x_{n-2}]) = n - 6$. Now it is easy to see that y_1, x_5 are non-adjacent (by Claim 12) and y_2, x_5 are adjacent (by Claim 13). Therefore, $d(y_1, C[x_6, x_{n-2}]) = n - 6$ and $y_1 x_6 \in D$ (by Lemma 3.2), $y_2 x_5, x_5 y_2 \in D$ (by Claim 11), $y_1 x_7 \notin D$ (by Claim 12). One readily sees that, by continuing the above procedure, we eventually obtain that n is even and

$$N^-(y_1) = \{y_2, x_2, x_4, x_6, \dots, x_{n-2}\}, \quad N^+(y_1) = \{y_2, x_4, x_6, \dots, x_{n-2}\},$$

$$N^-(y_2) = \{y_1, x_3, x_5, \dots, x_{n-3}\}, \quad N^+(y_2) = \{y_1, x_1, x_3, x_5, \dots, x_{n-3}\}.$$

From Claim 11 it follows that $x_i x_{i-1} \in D$ for all $i \in [4, n - 2]$ and $x_2 x_1 \in D$. It is easy to see that $x_1 x_3 \notin D$ and $x_3 x_5 \notin D$. Therefore, since x_3 cannot be inserted into $C[x_5, x_1]$, by Lemma 3.2, we have $d(x_3, C[x_5, x_1]) \leq n - 6$. This together with $d(x_3) = n$ (by Remark) implies that $d(x_3, \{x_2, x_4, y_2\}) = 6$. In particular, $x_3 x_2 \in D$. Now we consider the vertex x_{n-2} . Note that $d(x_{n-2}) = n$ (by Remark), $x_{n-2} x_2 \notin D$ and $x_{n-4} x_{n-2} \notin D$. From this it is not difficult to see that $d(x_{n-2}, C[x_2, x_{n-4}]) \leq n - 6$ and $x_1 x_{n-2} \in D$. It follows that $x_{n-2} x_{n-3} \dots x_4 x_3 y_2 x_1 x_{n-2}$ is a cycle of length $n - 2$, which does not contain the vertices y_1 and x_2 . This contradicts Claim 11, since $y_1 x_2 \notin D$ (by our supposition), i.e., $\langle \{y_1, x_2\} \rangle$ is not strong.

Assume next that $d^+(y_1, \{x_4, x_5, \dots, x_{n-2}\}) = 0$. Then from Claims 12 and 13 it follows that

$$N^-(y_1) = \{y_2, x_2, x_4, \dots, x_{n-2}\} \quad \text{and} \quad N^+(y_1) = \{y_2\}. \quad (19)$$

By Claim 14 we have that the vertex y_2 is adjacent with each vertex $x_i \in \{x_1, x_3, \dots, x_{n-3}\}$. It is easy to see that $x_{n-3} y_2 \notin D$ and hence, $y_2 x_{n-3} \in D$ (for otherwise if $x_{n-3} y_2 \in D$, then $y_2 C[x_1, x_{n-3}] y_2$ is a cycle of length $n - 2$, but $\langle \{x_{n-2}, y_1\} \rangle$ is not strong, a contradiction to Claim 11). By an argument similar to that in the proof of (19) we deduce that

$$N^+(y_2) = \{y_1, x_1, x_3, \dots, x_{n-3}\} \quad \text{and} \quad N^-(y_2) = \{y_1\}.$$

Thus we have that $y_1 y_2 C[x_5, x_2] y_1$ is a cycle of length $n - 2$ and x_3 cannot be inserted into $C[x_5, x_2]$. Therefore by Lemma 3.2(ii), $d(x_3, C[x_5, x_2]) \leq n - 4$ since $x_3 x_5 \notin D$. This together with $d(x_3, \{x_4, y_1, y_2\}) \leq 3$ implies that $d(x_3) \leq n - 1$ which contradicts the above Remark that $d(x_3) \geq n$.

Case 15.2. $y_2 x_3 \notin D$. Then, as noted above, $x_3 y_2 \in D$. Therefore $d(y_2, \{x_2, x_4\}) = 0$ (by Claim 12 and $y_2 x_2 \notin D$), $y_1 x_4 \notin D$ (by Claim 11), $x_4 y_1 \in D$ (by Claim 14), the vertices x_5, y_1 are non-adjacent

and the vertices y_2, x_5 are adjacent (by Claim 14). Since $x_3y_2 \in D$, $y_1x_4 \notin D$ and y_1, x_5 are adjacent, from Claim 11 it follows that $y_2x_5 \notin D$ and $x_5y_2 \in D$. For the same reason, we deduce that

$$N^-(y_1) = \{y_2, x_2, x_4, \dots, x_{n-2}\} \quad N^-(y_2) = \{y_1, x_1, x_3, \dots, x_{n-3}\} \quad \text{and} \quad A(R \rightarrow V(C)) = \emptyset,$$

which contradicts that D is strong. This contradiction completes the proof of Claim 15. \square

We will now complete the proof of Theorem by showing that D is isomorphic to $K_{n/2, n/2}^*$. Without loss of generality, we assume that $x_{n-2}y_1 \in D$. Then using Claims 11, 12, 13 and 15 we conclude that y_1, x_1 are non-adjacent (Claim 12), $y_1x_2 \in D$ (Claim 15), $x_1y_2, y_2x_1 \in D$ (Claim 11), x_2, y_2 also are non-adjacent (Claim 12) and $y_2x_3 \in D$ (Claim 15). By continuing these procedure, we eventually obtain that n is even and

$$N^+(y_1) = N^-(y_1) = \{y_2, x_2, x_4, \dots, x_{n-2}\} \quad \text{and} \quad N^+(y_2) = N^-(y_2) = \{y_1, x_1, x_3, \dots, x_{n-3}\}.$$

If $x_ix_j \in D$ for some $x_i, x_j \in \{x_1, x_3, \dots, x_{n-3}\}$, then clearly $|C[x_i, x_j]| \geq 5$ and $x_ix_jx_{j+1} \dots x_{i-1}y_1x_{i+1} \dots x_{j-2}y_2x_i$ is a cycle of length $n-1$, contrary to our assumption. Therefore $\{y_1, x_1, x_3, \dots, x_{n-3}\}$ is a independent set of vertices. For the same reason $\{y_2, x_2, x_4, \dots, x_{n-2}\}$ also is a independent set of vertices. Therefore D is isomorphic to $K_{n/2, n/2}^*$. This completes the proof of Theorem 1.10. \square

5 Concluding remarks

A Hamiltonian bypass in a digraph is a subdigraph obtained from a Hamiltonian cycle of D by reversing one arc.

Using Theorem 1.10, we have proved that if a strong digraph D of order $n \geq 4$ satisfies the condition A_0 , then D contains a Hamiltonian bypass or D is isomorphic to one tournament of order 5.

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